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Magnetostriction Transition

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We present a *non mean-field* model which undergoes a magnetostriction phase transition in the temperature. That is, the crystal becomes sharply contracted and magnetized once the temperature passes below the critical value.

KEY WORDS: Reflection positivity; first-order phase transitions; magnetostriction.

1. MODEL AND MAIN THEOREM 1

Magnetostriction is know in physics as a phenomenon of a drastic change of geometric shape of crystals, which is accompanied by magnetic transition, see, e.g., refs. 1 and 2. Usually it is a first order phase transition with a jump of spontaneous magnetization together with the jump in geometry of the crystal elementary cells. Physical origin of this phenomenon is related to so-called magnetoelastic coupling, i.e., to the interaction between spin and displacement degrees of freedom in magnetic crystals.[(1)] Various mean-field theories of this phenomenon were discussed in literature since a long time. See, e.g., ref. 3 and references therein, for crystals, and ref. 4 for magnetosriction in ferrofluids. (The solvable model with a short-range interaction, discussed in ref. 2, does not exhibit the jump specific for magnetostriction, because it is one-dimensional.)

In the present paper we propose a simple—and a first non mean-field!—model of this phenomenon. We prove that our model undergoes the phase transition, when the crystal becomes sharply contracted and magnetized, once the temperature passes below the critical value, provided the dimension is at least two.

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We consider the following model: at each site s of $\mathbb{Z}^d$ we have an Ising spin σ_s, while at each bond $l = \langle st \rangle$ of the lattice we have positive real variable r_{st}, playing the role of the spatial distance between two sites.

Initially we were interested in the Hamiltonian

$$\tilde{H}_\Lambda(\sigma^\Lambda, r^{\Lambda_b}) = -\sum_{\langle st \rangle \in \Lambda_b} J(r_{st})\, \sigma_s \sigma_t + \mu \sum_{\langle st \rangle \in \Lambda_b} (r_{st} - R)^2 - h \sum_{s \in \Lambda} \sigma_s.$$

Here the function $J(\cdot) \geqslant 0$ describes the dependence of the strength of the interaction between the spins σ_s and σ_t on their spatial separation. The parameter R is the ground-state distance between sites in the absence of the spin interaction; h is the external magnetic field. We were assuming that J is small on large distances and large on small distances. Our hope was to show that in the symmetric case—$h = 0$—the model would undergo the striction transition as the temperature goes down. But we were unable to show that, and, moreover, our computations suggest that such first order transition does not take place for the Hamiltonian $\tilde{H}$.

To realize our program we have to modify our Hamiltonian, adding another "geometric" term to the interaction. Namely, we will consider the model, defined by the following Hamiltonian:

$$H_\Lambda(\sigma^\Lambda, r^{\Lambda_b}) = -\sum_{\langle st \rangle \in \Lambda_b} J(r_{st})\, \sigma_s \sigma_t + \mu \sum_{\langle st \rangle \in \Lambda_b} (r_{st} - R)^2 + \lambda \sum_{\langle st \rangle,\, \langle s't \rangle \in \Lambda_b : |s - s'| = \sqrt{2}} (r_{st} - r_{s't})^2 - h \sum_{s \in \Lambda} \sigma_s. \tag{1}$$

Here in addition to the parameter $\mu > 0$, which is enforcing the lattice structure with the spacing to be close to R, we add another parameter $\lambda > 0$, which has the effect of making the r-lattice more regular. In particular, this term makes the "triangle inequality violation" energetically unfavourable. By the "triangle inequality violation" we mean, for example, the situation when among the four bonds r_{st}, $r_{s't}$, $r_{s't'}$, $r_{st'}$, forming a plaquette of the lattice, there are three relatively small values and one relatively big.

To ensure that the above model undergoes the striction transition we have to suppose that the interaction J is weak enough on large distances r, and is strong enough on small distances. Otherwise this function can be fairly general. We will describe now one specific choice of the class of interactions J, for which the transition takes place; other choices are also possible.

We are supposing that above some value of ρ, where $0 < \rho < R$, the interaction is weak:

$$J(r) \leqslant u \qquad \text{for} \quad r \geqslant \rho,$$

with u small. We further suppose that the interaction is bounded:

$$\max_{r>0} J(r) = \bar{U} < \infty,$$

and that within the region $r \leqslant \rho$ it is sufficiently strong: for some $K \subset [0, \rho]$ and for all $r \in K$

$$J(r) \geqslant U,$$

with U large, while $\frac{\bar{U}}{U} = 1 + \varkappa$ with $\varkappa$ small and meas$\{K\} \geqslant \rho/2$. As we show below (see (14)), the choice of the parameters R, ρ, U, u, and $\varkappa$ is possible, which guarantees the striction transition to happen.

The Hamiltonian (1) has the Reflection Positivity (RP) property with respect to reflections in the shifted coordinate planes:

$$L_{i;k} = \{x \in \mathbb{R}^d : x_i = k\}, \qquad i = 1,..., d,$$

with integer k; it is also RP with respect to reflections in the diagonal planes

$$L_{i,j;k} = \{x \in \mathbb{R}^d : x_i - x_j = k\}, \qquad i, j = 1,..., d, \quad i \neq j,$$

again for k integer. To simplify the computations we will use the latter; this, however, is applicable only in 2D case. The general case can also be treated, using the RP in coordinate planes, along the same lines.

To formulate our results, we introduce the indicators of some events: for a bond $l = st$ we define

$$P_l^<(\mathbf{r}, \sigma) = \begin{cases} 1 & \text{if} \quad r_{l=st} \leqslant \rho, \\ 0 & \text{otherwise,} \end{cases}$$

where $\rho > 0$ is a parameter to be chosen later. Similarly, we define the indicator

$$P_l^>(\mathbf{r}, \sigma) = \begin{cases} 1 & \text{if} \quad r_{l=st} \geqslant \rho + \varepsilon, \\ 0 & \text{otherwise,} \end{cases}$$

where $\varepsilon > 0$ is another parameter to be chosen.

We call a Gibbs state $\langle \cdot \rangle_\beta$, corresponding to the Hamiltonian (1) and inverse temperature β, a *contracted* state, iff for every l

$$\langle P_l^< \rangle_\beta \geqslant \tfrac{3}{4}.$$

Likewise, we call a Gibbs state $\langle\cdot\rangle_\beta$ an *expanded* state, iff for every l

$$\langle P_l^{>}\rangle_\beta \geqslant \tfrac{3}{4}.$$

Theorem 1. Let $h=0$. It is possible to choose the parameters of the Hamiltonian (1) in such a way, that the following holds :

- at all temperatures low enough there exists a contracted Gibbs state;
- at all temperatures high enough there exists an expanded Gibbs state;
- for some critical temperature β_c there exist at least two different Gibbs states, $\langle\cdot\rangle_{\beta_c}^{cn}$ and $\langle\cdot\rangle_{\beta_c}^{ex}$; the state $\langle\cdot\rangle_{\beta_c}^{cn}$ is contracted, while the state $\langle\cdot\rangle_{\beta_c}^{ex}$ is expanded;
- if there exists a contracted state at the temperature β^{-1}, then in fact there are at least two such states, $\langle\cdot\rangle_\beta^{+}$ and $\langle\cdot\rangle_\beta^{-}$. They are oppositely magnetized: for every s, t

$$\langle\sigma_s\rangle_\beta^{+} = -\langle\sigma_t\rangle_\beta^{-} \geqslant \tfrac{3}{4}.$$

Our result makes the following *conjectures* very plausible:

- above the critical temperature T_c every Gibbs state of our Hamiltonian is expanded, having zero magnetization,
- below T_c every Gibbs state is contracted, while every pure state has non-zero magnetization,
- at $T=T_c$ precisely three pure states coexist: one is expanded, with zero magnetization, while the other two are contracted and oppositely magnetized.

2. BASIC ESTIMATES AND PROOF OF THE MAIN RESULT

Our strategy of the proof is to follow the RP theory of the first-order phase transitions. To this end we introduce the following indicators:

$$\begin{aligned} &P_l^{<} \quad \text{–of the event} \quad \{r_{l=st} \leqslant \rho\},\\ &P_l^{0} \quad \text{–of the event} \quad \{\rho < r_{l=st} < \rho+\varepsilon\},\\ &P_l^{>} \quad \text{–of the event} \quad \{r_{l=st} \geqslant \rho+\varepsilon\},\end{aligned}$$

so $P_l^{<}+P_l^{0}+P_l^{>}=1$. We also introduce the indicators $P_\Lambda^{<}$, P_Λ^{0}, and $P_\Lambda^{>}$, which are products of the above over all bonds, i.e., $P_\Lambda^{<}=\prod_{l\in\Lambda_b} P_l^{<}$, etc. We put $P_\Lambda^{0>}=\prod_{l\in\Lambda_b}(P_l^{0}+P_l^{>})$.

The strategy consists in showing that for the finite volume states $\langle \cdot \rangle_\beta$ with periodic boundary conditions at inverse temperature β, uniformly in volume:

- the expectation $\langle P_l^> \rangle_\beta$ is small at low temperatures,
- the expectation $\langle P_l^< \rangle_\beta$ is small at high temperatures,
- the expectation $\langle P_l^< P_{l'}^> \rangle_\beta$ is small at all temperatures and for all pairs of bonds $l \neq l'$,
- the expectation $\langle P_l^0 \rangle_\beta$ is small at all temperatures.

The rest then is standard, see refs. 5 or 6.

(1) First we show that the expectation $\langle P_l^> \rangle_\beta$ is small at low temperatures.

$$\langle P_l^> \rangle_\beta \leqslant \langle P_\Lambda^> \rangle_\beta^{1/2\,|\Lambda|} \leqslant \left\{ \frac{\langle P_\Lambda^> \rangle_\beta}{\langle P_\Lambda^< \rangle_\beta} \right\}^{1/2\,|\Lambda|}.$$

We have :

$$\begin{aligned}\langle P_\Lambda^> \rangle_\beta &= \frac{1}{Z_\Lambda(\beta)} \sum_{\sigma^\Lambda} \int \prod_{l \in \Lambda_b} dr_l\, I_{\{r_l \geqslant \rho+\varepsilon\}}(r_l)\, e^{-\beta H_\Lambda(\sigma^\Lambda, r^{\Lambda_b})} \\ &\leqslant \frac{1}{Z_\Lambda(\beta)} \sum_{\sigma^\Lambda} \prod_{l \in \Lambda_b} \int_{\rho+\varepsilon}^{\infty} dr_l\, e^{-\beta\mu(r_l - R)^2 + \beta u} \leqslant \frac{1}{Z_\Lambda(\beta)}\, 2^{|\Lambda|} \left(\sqrt{\frac{\pi}{\beta\mu}} \right)^{2\,|\Lambda|} e^{2\,|\Lambda|\,\beta u}.\end{aligned}$$

Here and in the following we use the identity: $\int_{-\infty}^{+\infty} e^{-Ax^2}\, dx = \sqrt{\frac{\pi}{A}}$. On the other hand

$$\begin{aligned}\langle P_\Lambda^< \rangle_\beta &= \frac{1}{Z_\Lambda(\beta)} \sum_{\sigma^\Lambda} \int \prod_{l \in \Lambda_b} dr_l\, I_{\{r_l \leqslant \rho\}}(r_l)\, e^{-\beta H_\Lambda(\sigma^\Lambda, r^{\Lambda_b})} \\ &\geqslant \frac{1}{Z_\Lambda(\beta)}\, e^{2\beta U\,|\Lambda|} \prod_{l \in \Lambda_b} \int_K dr_l\, e^{-\beta\mu R^2} e^{-2\beta\lambda\rho^2} \\ &= \frac{1}{Z_\Lambda(\beta)} \left(\frac{\rho}{2}\, e^{\beta(U - \mu R^2 - 2\lambda\rho^2)} \right)^{2\,|\Lambda|}. \qquad (2)\end{aligned}$$

Therefore,

$$\langle P_l^> \rangle_\beta \leqslant \frac{2\sqrt{2\pi}\, e^{-\beta(U - u - \mu R^2 - 2\lambda\rho^2)}}{\rho\sqrt{\beta\mu}}, \qquad (3)$$

which is small for β large once

$$U-u>\mu R^2+2\lambda\rho^2. \tag{4}$$

(2) Next we show that the expectation $\langle P_l^{<}\rangle_\beta$ is small at high temperatures:

$$\langle P_l^{<}\rangle_\beta \leqslant \langle P_\Lambda^{<}\rangle_\beta^{1/2\,|\Lambda|} \leqslant \left\{\frac{\langle P_\Lambda^{<}\rangle_\beta}{\langle P_\Lambda^{0>}\rangle_\beta}\right\}^{1/2\,|\Lambda|}.$$

We have:

$$\begin{aligned}\langle P_\Lambda^{<}\rangle_\beta &= \frac{1}{Z_\Lambda(\beta)} \sum_{\sigma^\Lambda} \int \prod_{l\in\Lambda_b} dr_l\, I_{\{r_l\leqslant\rho\}}(r_l)\, e^{-\beta H_\Lambda(\sigma^\Lambda,\, r^{\Lambda_b})}\\ &\leqslant \frac{1}{Z_\Lambda(\beta)} (\sqrt{2}\,\rho e^{\beta(\bar{U}-\mu(\rho-R)^2)})^{2\,|\Lambda|}.\end{aligned}$$

For the lower bound we have

$$\begin{aligned}\langle P_\Lambda^{0>}\rangle_\beta &= \frac{1}{Z_\Lambda(\beta)} \sum_{\sigma^\Lambda} \int \prod_{l\in\Lambda_b} dr_l\, I_{\{r_l\geqslant\rho\}}(r_l)\, e^{-\beta H_\Lambda(\sigma^\Lambda,\, r^{\Lambda_b})}\\ &\geqslant \frac{1}{Z_\Lambda(\beta)}\, 2^{|\Lambda|} \int \prod_{l\in\Lambda_b} dr_l\, I_{\{r_l\geqslant\rho\}}(r_l)\, e^{-\beta(u+\mu(r_l-R)^2+\lambda\sum_{l':\, l\,\mathrm{nn}\, l'}(r_l-r_{l'})^2)}\\ &\geqslant \frac{1}{Z_\Lambda(\beta)}\, 2^{|\Lambda|} \prod_{l\in\Lambda_b} \int_{-(R-\rho)}^{\infty} dr_l\, e^{-\beta(\mu+8\lambda)\, r_l^2-\beta u}\\ &\geqslant \frac{1}{Z_\Lambda(\beta)} \left(\sqrt{\frac{\pi}{2\beta(\mu+8\lambda)}}\; e^{-\beta u}\right)^{2\,|\Lambda|},\end{aligned} \tag{5}$$

where we use in the third line the inequality $(x-y)^2\leqslant 2x^2+2y^2$, and also the fact that for every l the sum $\sum_{l':\, l\,\mathrm{nn}\, l'}(r_l-r_{l'})^2$ has 4 terms. Therefore

$$\langle P_l^{<}\rangle_\beta \leqslant 2\sqrt{\frac{\beta(\mu+8\lambda)}{\pi}}\;\rho e^{\beta(\bar{U}+u-\mu(\rho-R)^2)}, \tag{6}$$

which is small for small β.

(3) In order to show that the expectation $\langle P_l^{<} P_{l'}^{>}\rangle_\beta$ is small for all pairs of bonds $l\neq l'$, it is sufficient to estimate it only for pairs $l=\langle st\rangle$, $l'=\langle s't\rangle$, $|s-s'|=\sqrt{2}$, see ref. 5. We have:

$$\langle P_l^{<} P_{l'}^{>}\rangle_\beta \leqslant \langle P_\Lambda^{\gtrless}\rangle_\beta^{1/|\Lambda|},$$

where the indicator $P_\Lambda^{\gtrless}$ corresponds to the following event: on half of the bonds—$\Lambda_b^{\geqslant}$—of Λ_b—namely, on those which have one endpoint on the sublattice, generated by the vectors (1, 1) and (2, −2)—the event $r_. \geqslant \rho+\varepsilon$ happens, while on the remaining ones—$\Lambda_b^{\leqslant} = \Lambda_b \setminus \Lambda_b^{\geqslant}$—the event $r_. \leqslant \rho$ happens. Therefore

$$\begin{aligned}\langle P_\Lambda^{\gtrless}\rangle_\beta &= \frac{1}{Z_\Lambda(\beta)} \sum_{\sigma^\Lambda} \int e^{-\beta H_\Lambda(\sigma^\Lambda, r^{\Lambda_b})} \prod_{l \in \Lambda_b^{\geqslant}} \prod_{l' \in \Lambda_b^{\leqslant}} dr_l\, I_{\{r_l \geqslant \rho+\varepsilon\}}(r_l)\, dr_{l'}\, I_{\{r_{l'} \leqslant \rho\}}(r_{l'}) \\ &\leqslant \frac{1}{Z_\Lambda(\beta)} 2^{|\Lambda|} e^{\beta(\bar U + u - \mu(\rho-R)^2 - \lambda\varepsilon^2)\,|\Lambda|} \rho^{|\Lambda|} \left(\int_{\rho+\varepsilon}^\infty dr\, e^{-\beta\mu(r-R)^2} \right)^{|\Lambda|} \\ &\leqslant \frac{1}{Z_\Lambda(\beta)} \left(2\sqrt{\frac{\pi}{\beta\mu}}\, \rho e^{\beta(\bar U + u - \mu(\rho-R)^2 - \lambda\varepsilon^2)} \right)^{|\Lambda|}.\end{aligned}$$

To estimate the partition function from below we note that $P_\Lambda^{<}(\cdot) + P_\Lambda^{0>}(\cdot) \leqslant 1$, so

$$Z_\Lambda(\beta) \geqslant Z_\Lambda(\beta)(\langle P_\Lambda^{<}\rangle_\beta + \langle P_\Lambda^{0>}\rangle_\beta). \tag{7}$$

Using (2), (5), we thus have

$$\langle P_\Lambda^{\gtrless}\rangle_\beta \leqslant \frac{\left(2\sqrt{\frac{\pi}{\beta\mu}}\, \rho e^{\beta(\bar U + u - \mu(\rho-R)^2 - \lambda\varepsilon^2)} \right)^{|\Lambda|}}{\left(\frac{\rho}{2} e^{\beta(U - \mu R^2 - 2\lambda\rho^2)} \right)^{2\,|\Lambda|} + \left(\sqrt{\frac{\pi}{2\beta(\mu+8\lambda)}}\, e^{-\beta u} \right)^{2\,|\Lambda|}}. \tag{8}$$

By suppressing one of the terms in the denominator of (8) we get the following two estimates:

$$\langle P_l^{<} P_{l'}^{>}\rangle_\beta \leqslant \frac{8}{\rho}\sqrt{\frac{\pi}{\beta\mu}}\, e^{\beta(\bar U + u - 2U - \mu(\rho-R)^2 + 2\mu R^2 - \lambda\varepsilon^2 + 4\lambda\rho^2)}, \tag{9}$$

which is good for β large, and

$$\langle P_l^{<} P_{l'}^{>}\rangle_\beta \leqslant 4\sqrt{\frac{\beta}{\pi\mu}}\, \rho(\mu+8\lambda)\, e^{\beta(\bar U + 3u - \mu(\rho-R)^2 - \lambda\varepsilon^2)}, \tag{10}$$

which is good for β small. So we have to look for some intermediate value of β^*, such that for $\beta \geqslant \beta^*$ the rhs of (9) is small, while for $\beta \leqslant \beta^*$ the rhs of (10) is small. Of course, such value of the inverse temperature should be the one which makes the two terms in the denominator of (8) equal; in other words, the reasonable choice of the value β^* is to take it to be the solution of the equation

$$\frac{\rho}{2}\, e^{\beta(U-\mu R^2-2\lambda\rho^2)} = \sqrt{\frac{\pi}{2\beta(\mu+8\lambda)}}\, e^{-\beta u}. \tag{11}$$

But any choice of β^* would be as good as this one, provided only that the estimates (9) and (10) will turn into bounds small enough.

(4) The last estimate we need is that for the expectation $\langle P_l^0\rangle_\beta$. We have

$$\langle P_l^0\rangle_\beta \leqslant \langle P_\Lambda^0\rangle_\beta^{1/2\,|\Lambda|}.$$

Now

$$\begin{aligned}\langle P_\Lambda^0\rangle_\beta &= \frac{1}{Z_\Lambda(\beta)} \sum_{\sigma^\Lambda} \int \prod_{l\in\Lambda_b} dr_l\, I_{\{\rho<r_l<\rho+\varepsilon\}}(r_l)\, e^{-\beta H_\Lambda(\sigma^\Lambda, r^{\Lambda_b})}\\ &\leqslant \frac{1}{Z_\Lambda(\beta)}\, 2^{|\Lambda|} \left(\int_\rho^{\rho+\varepsilon} dr\, e^{-\beta[\mu(\rho+\varepsilon-R)^2-u]}\right)^{2\,|\Lambda|}\\ &\leqslant \frac{1}{Z_\Lambda(\beta)}\, (\sqrt{2}\,\varepsilon e^{-\beta[\mu(\rho+\varepsilon-R)^2-u]})^{2\,|\Lambda|}.\end{aligned}$$

Combining with the estimate (7) we find:

$$\langle P_\Lambda^0\rangle_\beta \leqslant \frac{(\sqrt{2}\,\varepsilon e^{-\beta[\mu(\rho+\varepsilon-R)^2-u]})^{2\,|\Lambda|}}{\left(\frac{\rho}{2}\, e^{\beta(U-\mu R^2-2\lambda\rho^2)}\right)^{2|\Lambda|} + \left(\sqrt{\frac{\pi}{2\beta(\mu+8\lambda)}}\; e^{-\beta u}\right)^{2\,|\Lambda|}}. \tag{12}$$

Here we can proceed as in the previous case, turning (12) into two different estimates, depending on the value of β. However, the case of the observable P_l^0 is easier, and it is sufficient to keep just one summand in the denominator of (12) in order to get a reasonable estimate on it. Namely, we keep the second one, arriving to

$$\langle P_l^0\rangle_\beta \leqslant 2\sqrt{\frac{\beta(\mu+8\lambda)}{\pi}}\; \varepsilon e^{-\beta[\mu(\rho+\varepsilon-R)^2-2u]}. \tag{13}$$

We now shall show that if we make for the Hamiltonian (1) the following choice of the interaction parameters :

$$\lambda=\mu=1,\quad U=2R^2,\quad \bar{U}=(2+\delta^2)\,R^2,\quad u=\delta,\quad \rho=R^{-1},\quad \varepsilon=2\delta R, \tag{14}$$

with R big enough and δ small enough, then all the expectations needed are small in the corresponding parts of the interval $[\beta_+, \beta_-]$, provided $\beta_+ = \beta_+(R, \delta)$ is small enough, and $\beta_- = \beta_-(R, \delta)$ is large enough.

Since the estimate (4) is satisfied under our choice (14), the relation (3) holds for all β large enough. As we said before, the rhs of (6) is small for all β small enough. Therefore it is enough to check that the rhs of (8) and (12) are small uniformly in all β.

To proceed with the estimate of the correlation function $\langle P_l^< P_{l'}^> \rangle_\beta$, as indicated above, we have to choose a value of the intermediate inverse temperature β^*. Our choice is

$$\beta^* = 2\, R^{-2} \ln R. \tag{15}$$

One can check that thus defined β^* is indeed an approximate solution to (11) *as* $R \to \infty$, though this is not important.

In the region $\beta \geqslant \beta^*$ we will use the estimate (9), which under the choice (14) becomes

$$\begin{aligned}
\langle P_l^< P_{l'}^> \rangle_\beta &\leqslant 8R \sqrt{\frac{\pi}{\beta^*}}\; e^{\beta^*((2+\delta^2)\,R^2 + \delta - 4R^2 - (R - R^{-1})^2 + 2R^2 - 4\delta^2 R^2 + 4R^{-2})} \\
&\leqslant 8R \sqrt{\frac{\pi}{\beta^*}}\; e^{\beta^*(-(1+2\delta^2)\,R^2)} \\
&\leqslant 8R^2 \sqrt{\frac{\pi}{2 \ln R}}\; R^{-2\,(1+2\delta^2)} \\
&\leqslant R^{-4\delta^2}
\end{aligned}$$

for R large.

In the region $\beta \leqslant \beta^*$ we shall use the estimate (10), which similarly becomes

$$\begin{aligned}
\langle P_l^< P_{l'}^> \rangle_\beta &\leqslant 36 \sqrt{\beta \pi^{-1}}\; R^{-1} e^{\beta((2+\delta^2)\,R^2 + 3\delta - (R - R^{-1})^2 - 4\delta^2 R^2)} \\
&\leqslant 36 \sqrt{\beta \pi^{-1}}\; R^{-1} e^{\beta(1-2\delta^2)\,R^2} \\
&\leqslant 36 \sqrt{\beta^* \pi^{-1}}\; R^{-1} e^{\beta^*(1-2\delta^2)\,R^2} \\
&\leqslant 36 \sqrt{2\pi^{-1} \ln R}\; R^{-4\delta^2} \\
&\leqslant R^{-3\delta^2}
\end{aligned}$$

for R large.

Finally we consider the bound (13), which becomes

$$\langle P_l^0 \rangle_\beta \leqslant 12 \sqrt{\frac{\beta}{\pi}}\, \delta R e^{-\beta[(R^{-1}+2\delta R-R)^2-2\delta]}$$

$$\leqslant 12 \sqrt{\frac{\beta}{\pi}}\, \delta R e^{-\beta(1-3\delta)^2 R^2}.$$

Note that the function $\sqrt{x}\, e^{-ax}$ has its maximum at $x=\frac{1}{2a}$, which equals to $\sqrt{\frac{1}{2ea}}$. Applying this to the last expression with $x=\beta R^2$, we get

$$\langle P_l^0 \rangle_\beta \leqslant 6\sqrt{2}\, \frac{\delta}{1-3\delta} \sqrt{\frac{1}{\pi e}},$$

which is small for small δ at any β.

3. MAGNETIZATION

Here we will prove the last statement of our theorem: the occurrence of spontaneous magnetization in the contracted states. To do this we split the event $\{r_{l=st} \leqslant \rho\}$ into four events, and we introduce the corresponding four indicators

$$P_l^{<\pm\pm} \quad \text{–of the event} \quad \{r_{l=st} \leqslant \rho, \sigma_s = \pm, \sigma_t = \pm\}.$$

We will show now that for all β the expectations $\langle P_l^{<+-} \rangle_\beta = \langle P_l^{<-+} \rangle_\beta$ are small, uniformly in the volume. Together with the obvious statements that

$$\langle P_l^{<++} \rangle_\beta = \langle P_l^{<--} \rangle_\beta$$

and

$$\langle P_l^{<++} \rangle_\beta + \langle P_l^{<--} \rangle_\beta + \langle P_l^{<+-} \rangle_\beta + \langle P_l^{<-+} \rangle_\beta = \langle P_l^{<} \rangle_\beta,$$

that implies our claim, due to the first part of our theorem and by subsequent application of the Theorem 4.1 of ref. 5.

We have

$$\langle P_l^{<-+} \rangle_\beta \leqslant \langle P_\Lambda^{<-+} \rangle_\beta^{1/2\,|\Lambda|},$$

where $P_{\Lambda}^{<-+}$ the indicator of the event that for every bond l' we have $r_{l'} \leqslant \rho$, while

$$\sigma_s = \begin{cases} +1 & \text{if } \; |s_1|+|s_2| \text{ is even,} \\ -1 & \text{otherwise.} \end{cases}$$

We denote this spin arrangement by $\sigma_{\pm}^{\Lambda}$. So

$$\begin{aligned} \langle P_{\Lambda}^{<-+} \rangle_{\beta} &= \frac{1}{Z_{\Lambda}(\beta)} \int e^{-\beta H_{\Lambda}(\sigma_{\pm}^{\Lambda}, r^{\Lambda_b})} \prod_{l \in \Lambda_b} dr_l \; I_{\{r_l \leqslant \rho\}}(r_l) \\ &\leqslant \frac{1}{Z_{\Lambda}(\beta)} (\rho e^{-\beta\mu(\rho-R)^2})^{2|\Lambda|}. \end{aligned}$$

As in (8)–(10), we have two estimates:

$$\langle P_l^{<-+} \rangle_{\beta} \leqslant \frac{\rho e^{-\beta\mu(\rho-R)^2}}{\frac{\rho}{2} e^{\beta(U-\mu R^2-2\lambda\rho^2)}} = 2e^{-\beta[U+\mu(\rho-R)^2-\mu R^2-2\lambda\rho^2]} \tag{16}$$

and

$$\langle P_l^{<-+} \rangle_{\beta} \leqslant \frac{\rho e^{-\beta\mu(\rho-R)^2}}{\sqrt{\frac{\pi}{2\beta(\mu+8\lambda)}}\, e^{-\beta u}} = \rho \sqrt{\frac{2\beta(\mu+8\lambda)}{\pi}}\, e^{-\beta[\mu(\rho-R)^2-u]}. \tag{17}$$

In fact, with our choice (14) of the parameters the second one is effective for all β. We have

$$\langle P_l^{<-+} \rangle_{\beta} \leqslant 3R^{-1} \sqrt{\frac{2\beta}{\pi}}\, e^{-\beta[(R-R^{-1})^2-\delta]}.$$

The rhs has its maximum at $\beta = \frac{1}{2[(R-R^{-1})^2-\delta]}$, so

$$\langle P_l^{<-+} \rangle_{\beta} \leqslant 3R^{-1} \sqrt{\frac{1}{\pi e[(R-R^{-1})^2-\delta]}},$$

which is small for R large enough. ▮

REFERENCES

1. C. Kittel, *Introduction to Solid State Physics*, 5th Ed. (Wiley, New York, 1997).
2. D. C. Mattis, *The Theory of Magnetism* (Harper & Row, New York, 1972).
3. V. A. Zagrebnov and V. K. Fedyanin, Spin-phonon interaction in the ising model, *Theor. Math. Phys.* **10**:127–142 (1972).
4. H.-O. Georgii and V. A. Zagrebnov, On the interplay of magnetic and molecular forces in curie-weiss ferrofluid model, *J. Stat. Phys.* **93**:79–107 (1998).
5. S. B. Shlosman, The method of reflection positivity in the mathematical theory of first-order phase transitions, *Russian Math. Surveys* **41**:83–134 (1986).
6. R. Kotecky and S. B. Shlosman, First-order phase transitions in large entropy lattice models, *Commun. Math. Phys.* **83**:493–515 (1982).